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Properties of equations of the continuous Toda type

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Abstract. We study a modified version of an equation of the continuous Toda type in $1+1$ dimensions. This equation contains a friction-like term which can be switched off by annihilating a free parameter ϵ. We apply the prolongation method, and the symmetry and approximate symmetry approaches. This strategy allows us to gain insight into both the equations for $\epsilon = 0$ and $\epsilon \neq 0$, whose properties arising from the above frameworks we compare. For $\epsilon = 0$, the related prolongation equations are solved by means of certain series expansions which lead to an infinite-dimensional Lie algebra. Furthermore, using a realization of the Lie algebra of the Euclidean group E_2, a connection is shown between the continuous Toda equation and a linear wave equation which resembles a special case of a three-dimensional wave equation that occurs in a generalized Gibbons–Hawking ansatz (Lebrun C 1991 *J. Diff. Geom.* **34** 223). Non-trivial solutions to the wave equation expressed in terms of Bessel functions are determined.

For $\epsilon \neq 0$, we obtain a finite-dimensional Lie algebra with four elements. A matrix representation of this algebra yields solutions of the modified continuous Toda equation associated with a reduced form of a perturbative Liouville equation. This result coincides with that achieved in the context of the approximate symmetry approach. Example of exact solutions are also provided. In particular, the inverse of the exponential-integral function turns out to be defined by the reduced differential equation arising from a linear combination of the time and space translations. Finally, a Lie algebra characterizing the approximate symmetries is discussed.

1. Introduction

We investigate the equation

$$u_{tt} + \epsilon u_t = (\mathrm{e}^u)_{xx} \qquad (1)$$

where $u = u(x, t)$, the subscripts denote partial derivatives and ϵ is a constant. For $\epsilon = 0$, equation (1) is a continuous Toda system in $1+1$ dimensions (or, equivalently, a two-dimensional version of the so-called heavenly equation: self-dual Einstein spaces with a rotational Killing vector [2]). The latter arises in many branches of physics, running from the theory of Hamiltonian systems, to the topological field theory [3]. In the case in which $\epsilon \neq 0$, the term ϵu_t mimics a friction-like behaviour. Equation (1) for $\epsilon \neq 0$ has been handled, in part, in [3] by means of an approximate group analysis.

Since 1989 up to now, there have been only a few applications of the Baikov–Gazizov–Ibragimov method [3]. This is a valid motivation to handle the continuous Toda equation with a perturbative term. In fact, the physical meaning and the 'ubiquitous' role of the Toda equation is well-established; the presence of a perturbative friction-like term is of interest

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in its own right, and characterizes the spirit of the paper both by a methodological and a speculative point of view.

We seek an algebraic characterization of (1) in both the cases $\epsilon = 0$ and $\epsilon \neq 0$. In doing so, we resort to the prolongation method [4] and the symmetry approach [5, 6]. For simplicity, we shall keep the formal machinery inherent in these techniques to a minimum. Our main results are as follows. For $\epsilon = 0$ equation (1) can be written as a set of (prolongation) differential equations which can be solved in terms of power-series expansions whose coefficients (in the variable $z = \mathrm{e}^u$) depend on the pseudopotential components and obey a presumably infinite-dimensional Lie algebra.

A remarkable fact is that this algebra can be closed 'step-by-step', in the sense that a finite-dimensional Lie algebra emerges corresponding to each polynomial in z coming from the truncation of the series. The use of a given representation of any closed Lie algebra allows us to find a linear problem associated with the equation under consideration. Furthermore, we show that the prolongation differential equations related to (1) ($\epsilon = 0$) afford a class of solutions connected to the Lie algebra of the Euclidean group E_2 in the plane. This enables us to map (1) to the linear wave equation

$$y_{tt} - \mathrm{e}^u y_{xx} = 0 \tag{2}$$

where $y = y(x, t)$ is a pseudopotential variable. We point out that equation (2) is equivalent to a $(1 + 1)$-dimensional version of [1, equation (2)] in which a generalized Gibbons–Hawking ansatz [7] pertinent to a quantum theory of gravity is considered. Starting from simple solutions of (1), examples of non-trivial solutions of (2) are shown.

In the case $\epsilon \neq 0$, the prolongation algebra for (1) turns out to be finite dimensional. This algebra is exploited to determine a class of special solutions via a reduced form of a Liouville-type equation. The latter coincides with that arising in the context of the symmetry approach. This feature suggests the existence of a possible link between the prolongation method and the symmetry approach, which deserves further study. Finally, a self-similar solution in terms of the exponential-integral function is obtained.

The paper is organized as follows. In section 2, the prolongation equations derived for (1) are studied. In sections 3 and 4, the cases $\epsilon = 0$ and $\epsilon \neq 0$ are considered, respectively. Section 5 deals with the symmetry and the approximate symmetry approach applied to (1). Precisely, the generators of the Lie point symmetries and approximate symmetries are found. The latter can be characterized by a finite-dimensional Lie algebra which admits a realization in terms of boson annihilation and creation operators. In section 6, the main results are summarized and some comments are made. Finally, appendices A and B contain details of calculations, while appendix C is devoted to a brief introduction to the approximate symmetry method.

2. The prolongation equations

Let us consider the prolongation system for (1):

$$y^i_x = F^i(u, u_t; y^j) \qquad y^i_t = G^i(u, u_x; y^j) \tag{3}$$

where $i, j = 1, 2, \ldots, N$ (N arbitrary), and the set of variables $\{y^i\}$ is the pseudopotential [4]. The compatibility condition for (3) yields

$$F^i = L^i u_t + M^i \qquad G^i = L^i \mathrm{e}^u u_x + P^i \tag{4}$$

where $M^i = M^i(u; y^j)$, $P^i = P^i(u; y^j)$, and $L^i = L^i(y^j)$ are defined by

$$M^i_u + [P, L]^i = \epsilon L^i \tag{5}$$

$$\mathrm{e}^u\,[L, M]^i = P^i_u \tag{6}$$

$$[M, P]^i = 0 \tag{7}$$

where $[P, L]^i = P^k\,\partial L^i/\partial y_k - L^k\,\partial P^i/\partial y_k$, and so on.

Hereafter we shall omit the index i for brevity. Now, we seek a solution of (5)–(7) of the form

$$M = \sum_{k=0}^{\infty} a_k(y)z^k \qquad P = \sum_{k=0}^{\infty} b_k(y)z^k \tag{8}$$

where $z = \mathrm{e}^u$ and y stands for the set of components $\{y^j\}$ $(j = 1, 2, \ldots, N)$.

By inserting the expansions (8) in (5)–(7), we obtain the following constraints between the coefficients $a_k(y)$ and $b_k(y)$:

$$[b_0, L] = \epsilon L \tag{9}$$

$$[L, b_k] = ka_k \tag{10}$$

$$[L, a_{k-1}] = kb_k \tag{11}$$

$$[a_0, b_0] = 0 \tag{12}$$

$$[a_0, b_1] + [a_1, b_0] = 0 \tag{13}$$

$$[a_0, b_2] + [a_1, b_1] + [a_2, b_0] = 0 \tag{14}$$

....

$$\sum_{k=1}^{N} [a_{k-1}, b_{N-k}] = 0 \tag{15}$$

where $k = 1, 2, \ldots N$ (N arbitrary).

In order to scrutinize the commutation relations (9)–(15), two cases have to be distinguished, i.e. $\epsilon = 0$ and $\epsilon \neq 0$.

3. The case $\epsilon = 0$

Let us assume $\epsilon = 0$ in (9)–(15). Then, the systematic application of the Jacobi identity to the commutation relations (9)–(15) produces an arbitrary number of finite-dimensional Lie algebras with $2N + 1$ generators ($N = 1, 2, \ldots$), i.e. $a_0,\ a_1, \ldots, a_N,\ b_0,\ b_1 \ldots, b_N$, and L (N arbitrary).

Although at present a rigorous proof is not given, this statement can be checked heuristically 'step-by-step' in the sense explained below.

To this end, let us take (the first step) $a_j = 0,\ b_j = 0$, for $j = 1, 2, \ldots$. Thus, from equations (9)–(15) we obtain the Abelian Lie algebra

$$[a_0, b_0] = 0 \qquad [a_0, L] = 0 \qquad [b_0, L] = 0. \tag{16}$$

Any realization of this algebra corresponds to a solution to (5)–(7) of the type $M = a_0(y)$, $P = b_0(y)$ (with $\epsilon = 0$). In this case, the prolongation system (3)–(4) becomes

$$y_x = Lu_t + a_0 \qquad y_t = L\mathrm{e}^u u_x + b_0. \tag{17}$$

Of course, the compatibility condition for the system (17) is fulfilled if (16) holds.

Now, let us choose (the second step) $a_j = 0$, $b_j = 0$, for $j = 2, 3, \ldots$. From equations (9)–(15) we have (see appendix A)

$$
\begin{array}{llll}
[a_0, a_1] = 0 & [a_0, b_0] = 0 & [a_0, b_1] = 0 & [a_0, L] = -b_1 \\
[a_1, b_0] = 0 & [a_1, b_1] = 0 & [a_1, L] = 0 & [b_0, b_1] = 0 \\
[b_0, L] = 0 & [b_1, L] = -a_1. & &
\end{array}
\tag{18}
$$

Equations (18) represent a non-Abelian Lie algebra defined by the five elements a_0, a_1, b_0, b_1 and L. Any realization of the algebra (18) corresponds to a solution to (5)–(7) of the type

$$M(u, y) = a_0(y) + a_1(y)\mathrm{e}^u \tag{19}$$

$$P(u, y) = b_0(y) + b_1(y)\mathrm{e}^u. \tag{20}$$

The prolongation equations (3), (4) related to (19), (20) are

$$y_x = Lu_t + a_0 + a_1\mathrm{e}^u \qquad y_t = L\mathrm{e}^u u_x + b_0 + b_1\mathrm{e}^u. \tag{21}$$

The compatibility condition for this system is verified by the algebra (18).

Furthermore, as shown in appendix A, under the hypothesis $a_j = 0$, $b_j = 0$, for $j = 3, 4, \ldots$ (the third step), equations (9)–(14) give rise to the prolongation Lie algebra defined by the seven elements a_0, a_1, a_2, b_0, b_1, b_2, and L:

$$
\begin{array}{llll}
[a_0, a_1] = 0 & [a_0, a_2] = 0 & [a_0, b_0] = 0 & [a_0, b_1] = 0 \\
[a_0, b_2] = 0 & [a_0, L] = -b_1 & [a_1, a_2] = 0 & [a_1, b_0] = 0 \\
[a_1, b_1] = 0 & [a_1, b_2] = 0 & [a_1, L] = -2b_2 & [a_2, b_0] = 0 \\
[a_2, b_1] = 0 & [a_2, b_2] = 0 & [a_2, L] = 0 & [b_0, b_1] = 0 \\
[b_0, b_2] = 0 & [b_0, L] = 0 & [b_1, b_2] = 0 & \\
[b_1, L] = a_1 & [b_2, L] = -2a_2. & &
\end{array}
\tag{22}
$$

The prolongation equations (3)–(4) can be written as

$$y_x = Lu_t + a_0 + a_1\mathrm{e}^u + a_2\mathrm{e}^{2u} \qquad y_t = L\mathrm{e}^u u_x + b_0 + b_1\mathrm{e}^u + b_2\mathrm{e}^{2u} \tag{23}$$

whose compatibility condition is ensured by (22).

The next step is to find a closed Lie algebra defined by the nine generators a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3, and L (see appendix A). Since the character of the commutation relations (A1)–(A12) is basically of the recursive type, we expect that the procedure of building up finite-dimensional Lie algebras starting from (9)–(15) ($\epsilon = 0$) works out for any step. In other words, we have the following possible scenario: equation (1) ($\epsilon = 0$) admits the prolongation Lie algebra defined by the commutation relations (see appendix A):

$$
\begin{array}{lll}
[L, a_{l-1}] = lb_l & [L, a_N] = 0 & [L, b_k] = ka_k \\
[a_j, a_k] = 0 & [a_j, b_k] = 0 & [b_j, b_k] = 0
\end{array}
\tag{24}
$$

for $j, k, = 0, 1, 2, \ldots, N$, and $l = 1, 2, \ldots, N$. Since N is an arbitrary positive integer, equations (24) represent an infinite-dimensional Lie algebra.

Another interesting result concerning the case $\epsilon = 0$, is expressed by the following proposition.

Proposition 1. Let u be a solution of the equation

$$u_{tt} = (\mathrm{e}^u)_{xx}. \tag{25}$$

Then, the function $y_2 = y_2(x, t)$ defined by

$$y_{2x} = -\mathrm{i}\mathcal{C}_0(\xi)\cosh y_1 \tag{26}$$

$$y_{2t} = -\tfrac{1}{2}\xi\mathcal{C}_1(\xi)\sinh y_1 \tag{27}$$

$$y_{1x} = u_t \qquad y_{1t} = \mathrm{e}^u u_x \tag{28}$$

satisfies the wave equation

$$y_{2tt} - \mathrm{e}^u y_{2xx} = 0 \tag{29}$$

where $\mathcal{C}_0(\xi)$, with $\xi = 2\mathrm{i}\mathrm{e}^{u/2}$, fulfills the differential equation of Bessel type

$$\frac{\mathrm{d}^2\mathcal{C}_0}{\mathrm{d}\xi^2} + \frac{1}{\xi}\frac{\mathrm{d}\mathcal{C}_0}{\mathrm{d}\xi} + \mathcal{C}_0 = 0 \tag{30}$$

and

$$\mathcal{C}_1 = \mathrm{i}\frac{\mathrm{d}\mathcal{C}_0}{\mathrm{d}\xi}. \tag{31}$$

To prove this proposition, let us search for a solution to (5)–(7) ($\epsilon = 0$) of the form

$$M(u, y) = m(u)g(y) \qquad P(u, y) = p(u)h(y). \tag{32}$$

Substituting equation (32) in (5)–(7) yields

$$m_u = p \qquad m\mathrm{e}^u = p_u \tag{33}$$

and

$$[L, h] = g \qquad [L, g] = h \qquad [g, h] = 0. \tag{34}$$

Equations (33) provide

$$m_{uu} - m\mathrm{e}^u = 0 \qquad p_{uu} - p_u = \mathrm{e}^u p \tag{35}$$

which give

$$m = \mathcal{C}_0(\xi) \qquad p = -\tfrac{1}{2}\mathrm{i}\xi\mathcal{C}_1(\xi) \tag{36}$$

respectively, where $\xi = 2\mathrm{i}\mathrm{e}^{u/2}$, and $\mathcal{C}$ denotes a function of Bessel type J, Y, $H^{(1)}$, $H^{(2)}$, or any linear combination of them [8].

We note that by setting $L = \mathrm{i}X_1$, $h = X_2$ and $g = \mathrm{i}X_3$, equations (34) become:

$$[X_1, X_2] = X_3 \qquad [X_1, X_3] = X_2 \qquad [X_2, X_3] = 0 \tag{37}$$

i.e. the Lie algebra associated with the Euclidean group, E_2, in the plane [9].

A realization of equations (37) in terms of a two-component pseudopotential $y = (y_1, y_2)$ is

$$X_1 = -\mathrm{i}\partial_{y_1} \qquad X_2 = -\mathrm{i}\sinh y_1\partial_{y_2} \qquad X_3 = -\cosh y_1\partial_{y_2}. \tag{38}$$

Then, equations (3) can be written (see equations (4) and (3)) as

$$y_{1x} = u_t \qquad y_{1t} = \mathrm{e}^u u_x \tag{39}$$

$$y_{2x} = -\mathrm{i}m\cosh y_1 \qquad y_{2t} = -\mathrm{i}p\sinh y_1. \tag{40}$$

Thus, equations (26)–(27) are simply equations (40) with m and p replaced by the quantities (36). Furthermore, equation (29) arises straightforwardly from equations (40) with the help of equations (33). Finally, equations (30) and (31) are a direct consequence of (4) and (9)–(15).

We observe also that by combining (29) and (25) we can express equation (25) by means of the pseudopotential y_2, namely

$$\partial_t^2 \ln \frac{y_{2tt}}{y_{2xx}} = \partial_x^2 \frac{y_{2tt}}{y_{2xx}}. \tag{41}$$

Below we shall display a few examples of non-trivial solutions of the wave equation (29), starting from some simple solutions of (25).

To this end, let us consider the simple solution $u = t$ to equation (25). Then, by choosing $\mathcal{C}_0(\xi) = \mathrm{i}J_0(\xi)$, so that $\mathcal{C}_1(\xi) = -\mathrm{d}J_0(\xi)/\mathrm{d}\xi = J_1(\xi)$ (see equation (36)), where J_0 and J_1 are Bessel functions of the first kind [8], from equation (40) we obtain

$$y_{2x} = J_0(\xi)\cosh x \tag{42}$$

$$y_{2t} = -\tfrac{1}{2}\xi J_1(\xi)\sinh x \tag{43}$$

where $\xi = 2\mathrm{i}\mathrm{e}^{t/2}$ ($y_{1x} = 1,\quad y_{1t} = 0$; see equation (39)).

Equations (42), (43) can easily be integrated to give

$$y_2(x,t) = J_0(2\mathrm{i}\mathrm{e}^{t/2})\sinh x. \tag{44}$$

Hence, the pseudopotential variable (44) represents a particular solution to the wave equation

$$y_{2tt} - \mathrm{e}^t y_{2xx} = 0. \tag{45}$$

Another application of proposition 1 concerns the equation

$$y_{2tt} - x y_{2xx} = 0 \tag{46}$$

which corresponds to $u = \ln x$ (a special solution to (25)). Using the same procedure as before, after staightforward calculations we find

$$y_2 = \frac{1}{2}\xi \frac{\mathrm{d}J_0(\xi)}{\mathrm{d}\xi}\cosh t \tag{47}$$

where $\xi = 2\mathrm{i}x^{\frac{1}{2}}$.

At this point it is instructive to show that the explicit form of the pseudopotential can be used to solve certain linear second-order ordinary differential equations with variable coefficients. In fact, by way of example, let us set

$$y_2 = f(x)g(t) \tag{48}$$

in equation (46). Then, equation (46) entails

$$g_{tt} - g = 0 \tag{49}$$

$$f_{xx} - \frac{1}{x} f = 0. \tag{50}$$

Now, since $g = \cosh t$ is a particular integral of (48), from (47) and (48) we obtain

$$f(x) = -\mathrm{i}x^{\frac{1}{2}} J_1(2\mathrm{i}x^{\frac{1}{2}}). \tag{51}$$

4. The case $\epsilon \neq 0$

For $\epsilon \neq 0$, the prolongation algebra of (1) is a Lie algebra $\mathcal{L}$ closed at the beginning (see appendix B). This reads

$$[a_0, b_0] = [a_0, b_1] = [L, b_1] = 0 \tag{52}$$

$$[b_0, b_1] = \epsilon b_1 \tag{53}$$

$$[b_0, L] = \epsilon L \tag{54}$$

$$[L, a_0] = b_1. \tag{55}$$

A matrix representation of $\mathcal{L}$ is

$$\begin{aligned} a_0 &= \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad b_0 = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ b_1 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{56}$$

In view of (56), equations (3) take the form

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_x = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & u_t & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \tag{57}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_t = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ \mathrm{e}^u & \mathrm{e}^u u_x & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \tag{58}$$

from which $y_1 = \lambda_1 \,\mathrm{e}^{\epsilon t}$, $y_2 = \zeta \,\mathrm{e}^{\epsilon t}$ and

$$y_{3\zeta} = -\frac{1}{\lambda_1} \,\mathrm{e}^{\epsilon t}\, u_t\, \zeta \tag{59}$$

$$y_{3t} = -\lambda_1 \,\mathrm{e}^{\epsilon t}\, \zeta^2 \frac{\partial}{\partial \zeta} \frac{\mathrm{e}^u}{\zeta} \tag{60}$$

where $\zeta = -\lambda_1 x + \lambda_2$, and λ_1, λ_2 are constants of integration. Here y_3 plays the role of a potential variable.

We remark that equations (59), (60) can be exploited to determine explicit solutions of (1) ($\epsilon \neq 0$). In doing so, let us seek, for instance, solutions of the type $u_t = \gamma(t)$. After some manipulations, equations (59) and (60) provide

$$u = \ln\left[\frac{1}{2\lambda_1^2}(\gamma_t + \epsilon\gamma)\zeta^2 + \alpha\zeta + \mathrm{e}^{-\epsilon t}\Gamma_t\right] \tag{61}$$

where $\gamma = \gamma(t)$, $\alpha = \alpha(t) = \mathrm{e}^{\beta}$ and $\Gamma = \Gamma(t)$ are such that

$$\gamma = \frac{\alpha_t}{\alpha} \tag{62}$$

$$\beta_{tt} + \epsilon\beta_t = c_1 \mathrm{e}^{\beta} \tag{63}$$

$$\Gamma_t = c_2\, \mathrm{e}^{\int \mathrm{d}t\, (\gamma + \epsilon)} \tag{64}$$

where c_1, c_2 are constants of integration.

It is noteworthy that equation (63), which is a modified version of the reduced Liouville equation, has the same form as that obtained in the framework of the Lie group approach [5] via a certain symmetry variable (see section 5).

In the simple case $c_1 = 0$, equation (63) can easily be solved. Then we find the exact solution to (1)

$$u = k_1 e^{-\epsilon t} + \ln\left(e^{k_2/\epsilon}\,\zeta + k_0\right) \tag{65}$$

where k_0, k_1 and k_2 are arbitrary constants. Otherwise (when $c_1 \neq 0$ and ϵ is a small parameter), equation (63) can be analysed by using some perturbative technique.

5. The symmetry approach

5.1. Symmetry generators

As is well known [6], in the study of a system of partial differential equations one can use symmetry groups to find special solutions (which are invariants under some subgroups of the complete symmetry group) by solving reduced systems of differential equations involving fewer independent variables than the original system. However, the standard technique has to be modified if small perturbations are present in the equations under consideration. In this context, the authors of [2] devised a method based on the concepts of an approximate group of transformations and approximate symmetries. For the reader's convenience, in appendix C we shall recall the main aspects of this method.

In order to obtain the approximate symmetries admitted by (1) ($\epsilon \neq 0$), in equations (C20) and (C21) let us take

$$\begin{aligned} &z = (t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) \qquad k = 1, 2, \ldots, 8 \\ &X = (\xi_0^1 + \epsilon\,\xi_1^1)\partial_t + (\xi_0^2 + \epsilon\,\xi_1^2)\partial_x + (\xi_0^3 + \epsilon\,\xi_1^3)\partial_u \end{aligned} \tag{66}$$

$$\xi^j = (\xi_o^3)^j + \epsilon(\xi_1^3)^j \qquad\qquad j = 4, \ldots, 8 \tag{67}$$

where

$$\begin{aligned} &(\xi_\alpha^3)^J = D_J\left(\xi_\alpha^3 - \sum_{i=1}^{2} \xi_\alpha^i\, u_i^\alpha\right) + \sum_{i=1}^{2} \xi_\alpha^i\, u_{J,i}^\alpha \\ &u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i} \qquad u_{J,i}^\alpha = \frac{\partial u_J^\alpha}{\partial x^i} \qquad \alpha = 0, 1 \end{aligned} \tag{68}$$

D_J denotes the total derivative with respect to $J = t, x, tt, tx, xx$, respectively, and $x^1 \equiv t$, $x^2 \equiv x$.

By making equations (C20) and (C21) explicit and keeping in mind (66)–(68), setting the coefficients of the independent variables $u_{0t},\ u_{0t}u_{0x}, \ldots$ to zero, we arrive at the following set of constraints:

$$\begin{aligned} &\xi_j^3 + 2\xi_{jt}^1 - 2\xi_{jx}^2 = 0 \qquad \xi_{jxx}^2 - 2\xi_{jx}^2 = 0 \\ &2\xi_{jtu}^3 - \xi_{jtt}^1 = 0 \qquad \xi_{jtt}^3 - e^{u_0}\xi_{jxx}^3 = 0 \qquad (j = 0, 1) \end{aligned} \tag{69}$$

and

$$2\xi_{jtu}^3 - \xi_{jtt}^1 + c_j = 0 \tag{70}$$

where $\xi_j^1 = \xi_j^1(t)$, $\xi_j^2 = \xi_j^2(x)$, $\xi_j^3 = \xi_j^3(x, t)$, and c_j is a constant of integration.

By solving equations (69) and (70), expression (66) takes the form

$$\begin{aligned} X = & \left[c_1 t + c_2 + \epsilon\left(\tfrac{1}{2}c_1 t^2 + k_1 t + k_2\right)\right]\partial_t \\ & + \left[(c_1 + c_3)x + c_4 + \epsilon\left(\tfrac{1}{2}k_3 + k_1\right)x + \epsilon k_4\right]\partial_x \\ & + \left[2c_3 + \epsilon(-2c_1 t + k_3)\right]\partial_u \end{aligned} \tag{71}$$

where $c_1,\ c_2, \ldots$ and $k_1,\ k_2, \ldots$ are arbitrary constants.

From the quantity (71) we have the operators

$$\begin{aligned} & X_1 = X_1^0 + \epsilon(\tfrac{1}{2}t^2\partial_t - 2t\partial_u) \qquad X_2 \equiv X_2^0 = \partial_t \\ & X_3 \equiv X_3^0 = \partial_x \qquad X_4 \equiv X_4^0 = x\partial_x + 2\partial_u \end{aligned} \tag{72}$$

where

$$X_1^0 = t\partial_t + x\partial_x \tag{73}$$

and

$$\begin{aligned} & X_5 = \epsilon(t\partial_t + x\partial_x) \qquad X_6 = \epsilon\partial_t \\ & X_7 = \tfrac{1}{2}\epsilon(x\partial_x + 2\partial_u) \qquad X_8 = \epsilon\partial_x. \end{aligned} \tag{74}$$

The operators X_1^0, X_2^0, X_3^0, X_4^0 are the (exact) symmetry generators relative to (1) for $\epsilon = 0$, while X_1, X_2, X_3, X_4 are the *approximate* generators of (1) for $\epsilon \neq 0$. The *exact* symmetry generators of (1)) for $\epsilon \neq 0$ are X_1^0, X_2^0 and X_3^0. The operators (74) are inessential, in the sense that ϵ is a constant factor.

5.2. Algebraic properties

The operators (72), (73) obey the commutation relations

$$[X_2, X_1] = X_2 + \epsilon(t\partial_t - 2\partial_u) \tag{75}$$

$$[X_2, X_4] = 0 \qquad [X_2, X_3] = 0 \quad [X_1, X_4] = 0 \tag{76}$$

$$[X_1, X_3] = -X_3 \qquad [X_3, X_4] = X_3. \tag{77}$$

We note that the commutation rules (75)–(77) do not define a (finite) Lie algebra. However, they can be used to build up a realization of a finite-dimensional Lie algebra by introducing the ‘auxiliary’ operators

$$Z = t\partial_t - 2\partial_u \qquad Y = \tfrac{1}{2}t^2\partial_t - 2t\partial_u. \tag{78}$$

In doing so, it turns out that $Y,\ Z,\ X_j\ (j = 1, \ldots, 4)$ satisfy the commutation relations

$$[X_2, X_1] = X_2 + \epsilon Z \qquad [X_3, X_1] = X_3 \qquad [X_1, X_4] = 0 \tag{79}$$

$$[X_1, Y] = Y \qquad [X_1, Z] = -\epsilon Y \tag{80}$$

$$[X_2, Y] = Z \qquad [X_2, Z] = X_2 \qquad [Z, Y] = Y \tag{81}$$

$$\begin{aligned} & [X_2, X_3] = [X_2, X_4] = [X_3, X_4] = [X_3, Y] = 0 \\ & [X_4, Y] = [X_3, Z] = [X_4, Z] = 0. \end{aligned} \tag{82}$$

For brevity, by the symbols $Y,\ Z,\ X_j\ (j = 1, \ldots, 4)$ we shall indicate both the abstract elements and the corresponding realization (72)–(73) and (75)–(77) of the finite-dimensional Lie algebra (79)–(82).

Now, let us focus our attention on the subalgebra (81). This is isomorphic to the $sl(2, R)$ algebra given by

$$[Z', T] = 2S \qquad [T, S] = 2Z' \qquad [S, Z'] = -2T \tag{83}$$

where

$$T = \sqrt{2}(Y + X_2) \qquad S = \sqrt{2}(Y - X_2) \qquad Z' = 2Z. \tag{84}$$

Furthermore, the following propositions hold.

Proposition 2. The Casimir operator

$$C = T^2 - S^2 - Z'^2 \tag{85}$$

$$\equiv 4[2X_2Y - Z(Z+1)] \tag{86}$$

of the Lie algebra (83), commutes with *all* the generators Y, Z, X_j $(j = 1, \dots, 4)$ of the Lie algebra (79)–(82).

The proof is straightforward.

Proposition 3. The algebra (79)–(82) admits a realization in terms of boson annihilation and creation operators.

This can be shown by setting

$$a_1^\dagger = t \qquad a_2^\dagger = u \qquad a_3^\dagger = x \qquad a_1 = \partial_t \qquad a_2 = \partial_u \qquad a_3 = \partial_x \tag{87}$$

to yield

$$[a_j, a_k^\dagger] = \delta_{jk} \qquad [a_j, a_k] = 0 \qquad [a_j^\dagger, a_k^\dagger] = 0 \qquad (j, k = 1, 2, 3) \tag{88}$$

$$Y = \tfrac{1}{2}a^{\dagger 2}_1 a_1 - 2a_1^\dagger a_2 \qquad Z = a_1^\dagger a_1 - 2a_2$$

$$X_1 = a_1^\dagger a_1 + a_3^\dagger a_3 + \epsilon\left(\tfrac{1}{2}a^{\dagger 2}_1 a_1 - 2a_1^\dagger a_2\right)$$

$$X_2 = a_1 \qquad X_3 = a_3 \qquad X_4 = a_3^\dagger a_3 + 2a_2 \tag{89}$$

and

$$C = -8a_2(2a_2 + 1). \tag{90}$$

We would like to complete this subsection with a remark on the ‘auxiliary’ operators (78). These are essential for establishing the closed algebra (79)–(82). Notwithstanding, their meaning is not completely clear. For instance it should be important to ascertain whether finite-dimensional Lie algebra analogous to (79)–(82) can be constructed in relation to other case studies. At present, we are able to affirm only that the operators (78), Z and Y can respectively be interpreted as symmetry variables of the equations

$$u_{xx} + \epsilon u_x = -(\mathrm{e}^{-u})_{tt} \tag{91}$$

$$u_{xx} + \epsilon u_x = -2(\mathrm{e}^{-u/2})_{tt} \tag{92}$$

which arise formally from (1) via the transformations $t \to x$, $u \to -u$, and $t \to x$, $u \to -u/2$. From Z we obtain the invariant $\eta(x) \equiv \mathrm{e}^{q(x)} = t\mathrm{e}^{u/2}$, that when inserted in (91) yields the reduced modified Liouville-type equation

$$q_{xx} + \epsilon\, q_x = -\mathrm{e}^{-2q}. \tag{93}$$

In a similar way, from (92) we find

$$r_{xx} + \epsilon\, r_x = -4\mathrm{e}^{-r/2} \tag{94}$$

where the invariant $r(x) = u + 4\ln t$ associated with Y has been used.

5.3. Explicit solutions

To see how the symmetry approach works out, we shall deal with two significant examples. First, let us write down the group transformations related to the generator $X_4 \equiv X_4^0$ (see equation (72)), which is present in both the cases $\epsilon = 0$ and $\epsilon \neq 0$. We have

$$x' = e^{\lambda} x \qquad t' = t \qquad u' = u + 2\lambda \tag{95}$$

where λ is the group parameter.

From equation (95) we deduce the invariant

$$\rho = x' \, e^{-u'/2} = x \, e^{-u/2}. \tag{96}$$

Then, making use of (96), equation (1) provides the reduced equation

$$\rho_t^2 - \rho\rho_{tt} - \epsilon\rho\rho_t = 1 \tag{97}$$

which is transformed into the (ordinary) modified Liouville-type equation

$$w_{tt} + \epsilon w_t = -e^{-2w} \tag{98}$$

through the change of variable $\rho = e^w$. We note that by setting $w = -\beta$, equation (98) coincides with (63) for $c_1 = 2$.

This property may be interpreted as a hint of a possible connection between the prolongation algebra and the symmetry generators coming from the group analysis. This important aspect pertinent to the algebraic theory of nonlinear field equations is a challenging subject for further investigation.

For $\epsilon = 0$, equation (98) is solved by

$$w = \ln \frac{\cos \sqrt{c}(t - t_0)}{\sqrt{c}} \tag{99}$$

where c and t_0 are positive constants.

Therefore, from (96) we obtain the exact solution

$$u = 2 \ln \frac{\sqrt{c} x}{\cos \sqrt{c}(t - t_0)} \tag{100}$$

to (1) (with $\epsilon = 0$).

We point out that the function (100) can be derived from (59) and (60) as a special case. In fact by choosing $\epsilon = 0$, equations (57) and (58) yield

$$y_{3x} = (\lambda_2 - \lambda_1 x) u_t \qquad y_{3t} = e^u [\lambda_1 + (\lambda_2 - \lambda_1 x) u_x] \tag{101}$$

from which

$$e^u = \tfrac{1}{2} \gamma_t x^2 \tag{102}$$

where we have assumed $u_t = \gamma(t)$, $\lambda_2 = 0$, and the functions of integration have been taken equal to zero. From equation (102) we obtain $u_t = \gamma_{tt}/\gamma_t \equiv \gamma$, i.e. $\gamma(t) = 2\sqrt{c} \tan \sqrt{c}(t - t_0)$. Consequently, equation (102) reproduces just the solution (100).

This result is not surprising, since the requirement $u_t = \gamma(t)$ means that we single out a class of solutions to (1) ($\epsilon = 0$) corresponding to the symmetry reduction based on the invariant $\rho(t)$ associated with the generator X_4^0. The relation between $\rho(t)$ and $\gamma(t)$ is $-2\rho_t/\rho = \gamma$ (see equation (96)). Finally, the algebra (52)–(55) with $\epsilon = 0$ coincides with that derived from (18) for $a_1 = 0$ (see appendix A).

Second, let us deal with the vector field obtained from the linear combination of the generators $X_3 = X_3^0 = \partial_x$ and $X_2 = X_2^0 = \partial_t$:

$$V = v\partial_x + \partial_t \tag{103}$$

where v is a constant.

The group transformations read $x' = x + v\lambda$, $t' = t + \lambda$, which furnish the invariant

$$\sigma = x' - vt' = x - vt. \tag{104}$$

This quantity can be exploited to find a self-similar solution to (1). In fact, by inserting $u = u(\sigma)$ in (1) we obtain

$$(v^2 - \mathrm{e}^u)u_\sigma = \epsilon v u + c_0 \tag{105}$$

where c_0 is a constant of integration.

By setting

$$u = -\tau + \ln v^2 \tag{106}$$

and $c_0 = -2\epsilon v \ln v$, equation (105) provides

$$\int_0^\tau \frac{1 - \mathrm{e}^{-\tau'}}{\tau'}\,\mathrm{d}\tau' = \frac{\epsilon}{v}(\sigma - \sigma_0) \tag{107}$$

($\xi_0 =$ constant).

The left-hand side of (107) is the exponential-integral function [8]

$$\mathrm{Ein}(\tau) = \mathrm{E}_1(\tau) + \ln\tau + \gamma \tag{108}$$

where $\mathrm{E}_1(\tau) = \int_\tau^\infty \left(\mathrm{e}^{-\tau'}/\tau'\right)\mathrm{d}\tau'$ and γ is the Euler constant. Hence, from equations (106), (107) and (108) we have the self-similar solution

$$u = -\mathrm{Ein}^{-1}\left[\frac{\epsilon}{v}(\sigma - \sigma_0)\right] - \frac{c_0}{\epsilon v} \tag{109}$$

where $\mathrm{Ein}^{-1}(\cdot)$ denotes the inverse function of (108). We note that the reduced equation (105) can be considered as an ordinary differential equation which defines the special function $\mathrm{Ein}^{-1}(\cdot)$.

6. Conclusions

We have studied a modified version of a continuous Toda equation in $1 + 1$ dimensions.

We have jointly applied two procedures: the prolongation method and the symmetry approach, which are mostly based on the use of algebraic and group techniques. This strategy shows itself to be effective both from a conceptual point of view and for practical purposes, e.g., for the determination of exact solutions of the equations under consideration.

For $\epsilon = 0$, on the basis of heuristic arguments we have found that an infinite-dimensional prolongation Lie algebra may be associated with equation (1). This algebra can be closed step-by-step in the sense explained in section 3. In correspondence of any representation of a given finite-dimensional Lie algebra in terms of pseudopotential variables, one obtains a linear problem for (1). Moreover, a link is established between (1) and the wave equation (37). This connection derives from a special realization of the Lie algebra of the Euclidean group E_2 related to a class of solutions of the prolongation equations (5)–(7) ($\epsilon = 0$). It is noteworthy that non-trivial solutions to (37) expressed in terms of particular Bessel functions are determined. We also remark that explicit forms of the pseudopotential can be used to solve certain second-order ordinary differential equations with non-constant coefficients (see, for example, equation (58)).

For the case $\epsilon \neq 0$, we have shown that the prolongation algebra is finite-dimensional and is constituted by four elements. A matrix representation of this algebra (see equation (64)) allows us to write (1) in a potential form which leads to solutions associated with those admitted by a modified version of the reduced Liouville equation (71). This equation has the same form as that coming from the symmetry approach via the generator $X_4 \equiv X_4^0 = x\partial_x + 2\partial_u$. This property indicates the existence of a possible connection between the prolongation method and the symmetry approach. This is an important methodological aspect which deserves a wide investigation. Here we remark only that any approach to this problem should not ignore the contribution by Harrison and Estabrook [10], where the fundamental concepts of Cartan's theory of systems of partial differential equations are exploited to obtain the generators of their invariance groups (isogroups). Another interesting result achieved in the framework of the symmetry approach is given by equation (109), which tells us that the inverse of the exponential-integral function turns out to be defined by the reduced differential equation (105) corresponding to the generator $V = v\partial_x + \partial_t$.

We shall conclude with a few comments.

First, we have not tackled the problem of the complete integrability of equation (1) (for $\epsilon = 0$). In any case, as occurs for other nonlinear partial differential equations of physical interest, the existence of an infinite-dimensional prolongation algebra is necessary for the integrability property. However, in this regard, we recall that completely integrable nonlinear field equations admit Kac-Moody prolongation algebras endowed with a loop structure. Therefore, a definitive answer on the complete integrability of (1) ($\epsilon = 0$) is strictly connected with a deep study of its associated algebra (24). The situation is different for $\epsilon \neq 0$, in which a finite-dimensional prolongation algebra is found at the beginning. This feature indicates that equation (1) is not completely integrable for $\epsilon \neq 0$.

Second, in the context of the approximate symmetry approach, i.e. when ϵ is a small parameter, a realization of a finite-dimensional Lie algebra, i.e. equations (79)–(82), can be constructed by introducing the auxiliary operators (78). This realization can be expressed in terms of boson annihilation and creation operators. The algebra (79)–(82), in some sense, seems to characterize the approximate symmetry of (1), but its role is not yet clear. For instance, may algebras of this type arise in the study of the approximate symmetries of other perturbative systems? Finally, our results could be useful in the study of a three-dimensional extension of equation (1), using the same theoretical framework of this paper. Concerning this point it should be interesting to see, in analogy with what happens in our case, whether the equations of [1, proposition 1]

$$u_{xx} + u_{yy} + (\mathrm{e}^u)_{zz} = 0 \tag{110}$$

$$w_{xx} + w_{yy} + (we^u)_{zz} = 0 \tag{111}$$

share a property similar to that linking equations (25) and (29), through a representation of a Lie algebra of an extended Euclidean group. Third, we note that one expects for (1) the shock formation (solution breaking in finite time), and one should ask how this phenomenon could appear in the algebraic structure allowed by (1). The problem of shock formation for (1) with $\epsilon = 0$ was studied by Kodama within the scheme of the inverse scattering transform [11]. This author derived an explicit solution formula for the initial value problem and found that the general solution may break in finite time. It should be interesting to look for a possible connection between the Kodama approach and the algebraic strategy developed in this paper. However , this task is not so easy mainly because we have not solved a Cauchy problem, in the sense that we have not investigated how a given initial condition evolves. On the other hand, the algebraic properties of (1) are based on the ansatz (8), i.e. come from the assumption that M and P can be expanded in power series in the variable $z = \exp(u)$, and we have no proof, at present, that we can obtain the general solution of the prolongation equations (5), (6), and (7) in such a way. Consequently, the problem of relating the phenomenon of shock formation to the algebraic structure of equation (1) remains open.

Appendix A. Closed Lie algebras from equations (9)–(15) with $\epsilon = 0$

Exploiting the Jacobi identity, from equations (10) and (11) we obtain

$$[a_k, a_{k'}] = -\frac{1}{k'}\{[L, [b_{k'}, a_k]] - (k+1)[b_{k'}, b_{k+1}]\} \tag{A1}$$

$$[b_{k'}, a_k] = -\frac{1}{k}\left\{[L, [b_k, b_{k'}]] - k'[b_k, a_{k'}]\right\} \tag{A2}$$

$$[b_k, b_{k'}] = -\frac{1}{k'}\{[L, [a_{k'-1}, b_k]] - k[a_{k'-1}, a_k]\}. \tag{A3}$$

Equations (A2) and (A3) provide

$$[b_0, a_k] = \frac{1}{k}[L, [b_0, b_k]] \tag{A4}$$

$$[b_0, b_k] = -\frac{1}{k}[L, [a_{k-1}, b_0]] \tag{A5}$$

which give

$$[b_0, a_k] = \frac{1}{(k!)^2}[L, [L, \ldots, [L, [L, [b_0, a_1]] \cdots] \tag{A6}$$

and

$$[b_0, b_k] = \frac{1}{k[(k-1)!]^2}[L, [L, [b_0, a_1]] \cdots] \tag{A7}$$

where the operator L on the right-hand side of (A6) and (A7) appears $2(k-1)$ and $2(k-1)-1$ times, respectively.

Other useful relations are

$$[a_0, a_k] = -\frac{1}{k}\{[L, [b_k, a_0]] - (k+1)[b_k, b_1]\} \tag{A8}$$

$$[a_0, b_k] = \frac{1}{k}\{[L, [a_0, a_{k+1}]] + [a_{k-1}, b_1]\} \tag{A9}$$

$$[a_k, b_k] = \frac{1}{k}\{[L, [a_k, a_{k-1}]] + (k+1)[a_{k-1}, b_{k+1}], \} \tag{A10}$$

where

$$[a_k, a_{k-1}] = \frac{1}{k}[L, [b_k, a_{k-1}]]. \tag{A11}$$

We also have

$$[b_{k-1}, b_k] = \frac{1}{k}[L, [b_{k-1}, a_{k-1}]]. \tag{A12}$$

Now, let us suppose that $a_j = 0$, for $j = 2, 3, \ldots$. Then, from equations (9), (10) and (11) we obtain

$$[b_0, L] = 0 \qquad [L, a_0] = b_1 \qquad [L, b_1] = a_1. \tag{A13}$$

Furthermore, from equations (A4) and (13):

$$[b_0, a_1] = 0 \qquad [a_0, b_1] = 0 \tag{A14}$$

while

$$[a_0, a_1] = 0 \tag{A15}$$

from (A8). We also have

$$[b_0, b_1] = 0 \tag{A16}$$

by commuting equation (12) with L. Finally, (A10) provides

$$[a_1, b_1] = 0 \tag{A17}$$

since $b_2 = 0$. The previous commutation relations define the algebra (27).

Now, let us suppose that $a_j = 0$, $b_j = 0$, for $j = 3, 4, \ldots$. Then, equation (7) yields

$$[a_0, b_0] = 0 \tag{A18}$$

$$[a_0, b_1] + [a_1, b_0] = 0 \tag{A19}$$

$$[a_0, b_2] + [a_1, b_1] + [a_2, b_0] = 0 \tag{A20}$$

$$[a_1, b_2] + [a_2, b_1] = 0 \tag{A21}$$

$$[a_2, b_2] = 0. \tag{A22}$$

By commuting (A18) with L and using the Jacobi identity, with the help of (9) ($\epsilon = 0$) and (11) we find that

$$[b_0, b_1] = 0. \tag{A23}$$

Taking account of (A23), equation (A4) gives

$$[b_0, a_1] = 0. \tag{A24}$$

Hence, from equation (A19):

$$[a_0, b_1] = 0. \tag{A25}$$

Furthermore, equations (A11), (A25) and (A10) give

$$[a_1, a_0] = 0 \tag{A26}$$

$$[a_1, b_1] = 0. \tag{A27}$$

Since

$$[a_2, b_0] = 0 \tag{A28}$$

(see equations (A6) and (A24)), from equation (A19) we deduce

$$[a_0, b_2] = 0. \tag{A29}$$

On the other hand, from equations (A3) (see equation (A27)):

$$[b_1, b_2] = 0 \tag{A30}$$

while (A2) provides

$$[b_2, a_1] = 2[b_1, a_2]. \tag{A31}$$

Keeping in mind (A31), from equation (A21) we have

$$[b_1, a_2] = [b_2, a_1] = 0. \tag{A32}$$

We also obtain

$$[a_2, a_1] = 0 \tag{A33}$$

from (A1) and (A2). Finally, equation (A8) gives (see equations (A29) and (A30))

$$[a_0, a_2] = 0. \tag{A34}$$

Thus, all the commutators among the elements a_j, b_j (j = 0,1,2) have been determined. The commutators of the type $[a_j, L]$ and $[b_j, L]$ are expressed by (10) and (11). Therefore, we have the seven-dimensional Lie algebra (31).

Another example of finite-dimensional Lie algebra emerging from the commutation relations (9)–(15), involves the nine generators a_j, b_j and L with $j = 0, 1, 2, 3$. This algebra can be obtained following essentially the same scheme adopted for the construction of the algebra (31).

In fact, by setting

$$\begin{aligned} M &= a_0 + a_1 z + a_2 z^2 + a_3 z^3 \\ P &= b_0 + b_1 z + b_2 z^2 + b_3 z^3 \end{aligned} \tag{A35}$$

equation (7) entails

$$[a_0, b_0] = 0 \tag{A36}$$

$$[a_0, b_1] + [a_1, b_0] = 0 \tag{A37}$$

$$[a_0, b_2] + [a_1, b_1] + [a_2, b_0] = 0 \tag{A38}$$

$$[a_0, b_3] + [a_1, b_2] + [a_2, b_1] + [a_3, b_0] = 0 \tag{A39}$$

$$[a_1, b_3] + [a_2, b_2] + [a_3, b_1] = 0 \tag{A40}$$

$$[a_2, b_3] + [a_3, b_2] = 0 \tag{A41}$$

$$[a_3, b_3] = 0. \tag{A42}$$

Equations (A36)–(A38) have already been examined. We need to scrutinize only those relations which contain a_3 and b_3, i.e. equations (A39)–(A42).

Since

$$[a_3, b_0] = 0 \tag{A43}$$

(see equations (A6) and (A24)), equation (A39) becomes

$$[a_0, b_3] + [a_1, b_2] + [a_2, b_1] = 0. \tag{A44}$$

Now, by commuting (A34) and (A30), we easily find

$$3[a_0, b_3] = [a_2, b_1] \tag{A45}$$

$$2[b_1, a_2] = [b_2, a_1]. \tag{A46}$$

Combining equations (A44), (A45) and (A46) we arrive at

$$[a_0, b_3] = 0 \tag{A47}$$

$$[a_2, b_1] = 0 \tag{A48}$$

$$[a_1, b_2] = 0. \tag{A49}$$

From equations (A8) and (A47)

$$[a_0, a_3] = \tfrac{1}{3}[b_3, b_1]. \tag{A50}$$

On the other hand, equations (A47) and (A49) give

$$[a_1, a_2] = 0. \tag{A51}$$

Thus, using (A3), (A48) and (A51), we have

$$[b_1, b_3] = 0. \tag{A52}$$

Hence, from equation (A50)

$$[a_0, a_3] = 0. \tag{A53}$$

At this stage, let us consider relation (A10) for $k = 2$. We have

$$[b_2, a_2] = \tfrac{3}{2}[a_1, b_3]. \tag{A54}$$

Furthermore, from equations (A2) and (A52)

$$[b_3, a_1] = 3[b_1, a_3]. \tag{A55}$$

Now, by elaborating the commutator $[L, [a_0, a_3]]$ by means of the Jacobi identity (see equation (A53)) and taking b_4=0, we obtain

$$[a_3, b_1] = 0. \tag{A56}$$

Thus, equation (A55) implies

$$[b_3, a_1] = 0 \tag{A57}$$

so that (A54) yields

$$[a_2, b_2] = 0. \tag{A58}$$

Inserting (A58) in (A12) for $k = 3$, we have

$$[b_2, b_3] = 0. \tag{A59}$$

Then, from equation (A1):

$$[a_1, a_3] = 0 \tag{A60}$$

via (A52) and (A59).

In conclusion, equation (A52) provides

$$3[b_2, a_3] = 2[b_3, a_2] \tag{A61}$$

(see equation (A59)). Therefore, equations (A61) and (A41) furnish

$$[b_3, a_2] = 0 \tag{A62}$$

$$[b_2, a_3] = 0. \tag{A63}$$

We also have

$$[a_2, a_3] = 0 \tag{A64}$$

from (A1) and (A62). Therefore, by collecting all the results achieved in the case $a_j = 0$, $b_j = 0$ $(j = 4, 5, \ldots)$, the finite-dimensional Lie algebra of elements a_0, a_1, $a_2\, a_3$, b_0, b_1, b_2, b_3 and L turns out to be defined by the commutation relations

$$\begin{aligned} &[a_j, a_k] = 0 \qquad [a_j, b_k] = 0 \qquad [b_j, b_k] = 0 \\ &[L, a_{l-1}] = l b_l \qquad [L, a_3] = 0 \qquad [L, b_k] = k a_k \end{aligned} \tag{A65}$$

for $j, k = 0, 1, 2, 3$ and $l = 1, 2, 3$.

Appendix B. Finite-dimensional prolongation algebra ($\epsilon \neq 0$)

Here we prove that the prolongation algebra associated with (1) with $\epsilon \neq 0$ is the finite-dimensional Lie algebra (52)–(55). In doing so, let us start from the commutation rule

$$[L, b_1] = a_1 \tag{B1}$$

coming from (10) for $k = 1$.

Then, by exploiting the Jacobi identity we have

$$[L, [b_0, b_1]] + \epsilon a_1 = [b_0, a_1] \tag{B2}$$

with the help of (9). On the other hand, relation (11) (with $k = 1$) provides

$$[b_0, b_1] = \epsilon b_1 \tag{B3}$$

by elaborating the commutator $[b_0, [L, a_0]]$ via the Jacobi identity. Now, substitution of (B3) in (B2) gives

$$[b_0, a_1] = 2\epsilon a_1. \tag{B4}$$

Taking account of (B4), equation (13) yields

$$[a_0, b_1] = 2\epsilon a_1. \tag{B5}$$

At this point, let us consider the commutator $[b_1[a_0, b_0]]$ (see equation (12)), By resorting again to the Jacobi identity and using (B3), (B4) and (B5) we obtain the constraint

$$\epsilon^2 a_1 = 0. \tag{B6}$$

Let us suppose that $\epsilon \neq 0$. Hence, equation (B6) implies $a_1 \equiv 0$. Consequently, from (10) and (11) we infer that $2b_2 = [L, a_1] = 0$, $2a_2 = [L, b_2] = 0$, $3b_3 = [L, a_2] = 0$, $3a_3 = [L, b_3] = 0$, and so on. Therefore, for $\epsilon \neq 0$ the quantities a_0, a_1, b_0, b_1, and L, define the closed Lie algebra $\mathcal{L}$ expressed by (52)–(55).

Appendix C. The approximate Lie group analysis

Here we recall some basic concepts inherent in the approximate group analysis of differential equations derived by Baikov, Gazizov and Ibragimov (BGI) in 1988 [5]. The starting point of this approach is a theorem (see below) that allows one to construct approximate symmetries which are stable for small perturbations of the differential equation under investigation.

To describe the BGI method briefly let us consider the one-parameter group of local point transformations

$$z' = g(z, \epsilon, a) \tag{C1}$$

where $z = (z_1, \ldots, z_N)$ is the independent variable, and a is the group parameter so that the value $a = 0$ corresponds to the identity transformation $g(z, \epsilon, a) = z$,

$$g(g(z, \epsilon, a), \epsilon, b) = g(z, \epsilon, a + b) \tag{C2}$$

and ϵ is a perturbative parameter. Then let us suppose that $f \approx g$, namely

$$f(z, \epsilon, a) = g(z, \epsilon, a) + \mathrm{o}(\epsilon^p) \tag{C3}$$

for some fixed values of $p \geqslant 0$.

The transformations

$$z' \approx f(z, \epsilon, a) \tag{C4}$$

form an approximate one-parameter group f

$$f(z,\epsilon,0) \approx z \tag{C5}$$

$$f(f(z,\epsilon,a),\epsilon,b) \approx f(z,\epsilon,a+b) \tag{C6}$$

and the condition $f(z,\epsilon,a) \approx z$ for all z implies that a =0.

The following theorem holds.

Theorem 1. Let us assume that the transformations (C4) form an approximate group with the tangent vector field

$$\xi(z,\epsilon) \approx \left.\frac{\partial f(z,\epsilon,a)}{\partial a}\right|_{a=0}. \tag{C7}$$

Then, the function $f(z,\epsilon,a)$ satisfies

$$\frac{\partial f(z,\epsilon,a)}{\partial a} \approx \xi(f(z,\epsilon,a)). \tag{C8}$$

Conversely, for any (smooth) function $\xi(z,\epsilon)$ the solution (C4) of the approximate Cauchy problem

$$\begin{aligned}&\frac{\mathrm{d}z'}{\mathrm{d}a} \approx \xi(z',\epsilon)\\ &z'|_{a=0} \approx z\end{aligned} \tag{C9}$$

determines an approximate one-parameter group with parameter a.

Theorem 1 is called the *approximate Lie theorem*, while equation (C9) is called the *approximate Lie equation*.

Resorting to the approximate Lie theorem, we can construct the approximate group of transformations generated by a given infinitesimal operator. To see how the method works in practice, let us deal with the case $p = 1$. Then we seek the approximate group of transformations

$$z' \approx f_0(z,a) + \epsilon f_1(z,a) \tag{C10}$$

determined by the infinitesimal operator

$$X = (\xi_0(z) + \epsilon\xi_1(z))\frac{\partial}{\partial z}. \tag{C11}$$

The related approximate Lie equation

$$\frac{\mathrm{d}}{\mathrm{d}a}(f_0+\epsilon f_1) \approx \xi_0(f_0(z)+\epsilon f_1) + \epsilon\xi_1(f_0(z)+\epsilon f_1) \tag{C12}$$

becomes the system

$$\frac{\mathrm{d}f_0}{\mathrm{d}a} \approx \xi_0(f_0) \qquad \frac{\mathrm{d}f_1}{\mathrm{d}a} \approx \xi_0'(f_0)f_1 + \xi_1(f_0) \tag{C13}$$

where ξ_0' is the derivative of ξ_0. The initial condition $z'|_{a=0} \approx z$ provides $f_0|_{a=0} \approx z$ and $f_1|_{a=0} \approx 0$.

We are now ready to introduce the concept of approximate invariance.

Precisely, the approximate equation

$$F(z,\epsilon) \approx 0 \tag{C14}$$

is said to be invariant with respect to the approximate group of transformations

$$z' \approx f_0(z,\epsilon,a)$$

if

$$F(f(z,\epsilon,a)) \approx 0 \tag{C15}$$

for all z satisfying (C15). A criterion for obtaining the approximate symmetries of a given equation is expressed by the following theorem.

Theorem 2. Let us suppose that the function $F(z,\epsilon) = \big(F^1(z,\epsilon),\ldots,F^n(z,\epsilon)\big)$, $n < N$, which is jointly analytic in the variables z and ϵ, satisfies the condition

$$\operatorname{rank} F'(z,0)\big|_{F(z,0)=0} = n$$

where $F'(z,0) = \left\|\partial F^\nu(z,0)/\partial z^i\right\|$ for $\nu = 1,\ldots,n$ and $i = 1,\ldots,N$. For the equation

$$F(z,0) = \mathrm{o}(\epsilon^p) \tag{C16}$$

to be invariant under the approximate group of transformations

$$z' = f(z,\epsilon,a) + \mathrm{o}(\epsilon^p) \tag{C17}$$

with infinitesimal operator

$$X = \xi(z,\epsilon)\,\frac{\partial}{\partial z} \qquad \xi = \left.\frac{\partial f}{\partial a}\right|_{a=0} + \mathrm{o}(\epsilon^p) \tag{C18}$$

it is necessary and sufficient that

$$XF(z,\epsilon) = \mathrm{o}(\epsilon^p) \tag{C19}$$

whenever $F(z,\epsilon) = \mathrm{o}(\epsilon^p)$.

In order to solve (C18) to within $\mathrm{o}(\epsilon^p)$ one needs to represent z, F and ξ^k in the form

$$z \approx y_0 + \epsilon y_1 + \ldots + \epsilon^p y_p \qquad F(z,\epsilon) \approx \sum_{i=0}^{p} \epsilon^i F_i(z)$$

$$\xi^k(z,\epsilon) \approx \sum_{i=0}^{p} \epsilon^i \xi_i^k(z).$$

For $p = 1$, we obtain

$$\xi_0^k(y_0)\left.\frac{\partial F_0}{\partial z^k}\right|_{z=y_0} = 0 \tag{C20}$$

$$\xi_1^k(y_0)\left.\frac{\partial F_0}{\partial z^k}\right|_{z=y_0} + \xi_0^k(y_0)\left.\frac{\partial F_1}{\partial z^k}\right|_{z=y_0} + y_1^l\,\frac{\partial}{\partial z^l}\left.\left[\xi_0^k(z)\,\frac{\partial F_0}{\partial z^k}\right]\right|_{z=y_0} = 0 \tag{C21}$$

under the conditions

$$F_0(y_0) = 0 \qquad F_1(y_0) + y_1^l\left.\frac{\partial F_0(z)}{\partial z^l}\right|_{z=y_0} = 0.$$

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